**Chapter 13**

Linear Programming (LP):
Model Formulation & Graphical Solution

**Introduction**

- Have a deterministic setup
- Make decisions using LP methods with resource constraints
- Is LP computer programming?
  - NO!
  - Predetermined set of mathematical steps used to solve linear equations
- First step to solving LP problem is formulation of the model

**Components of LP Problem**

- **Decision Variables**
  - Denoted by mathematical symbols that does not have a specific value
  - Examples
    - How much of a resource to devote to a product
    - What investment strategy to use & how long
    - What people should do what tasks
    - Borrow money or sell off inventory

**Components of LP Problem (Cont.)**

- **Objective Function**
  - Describes the goal in terms of the decision variables
  - Maximize or Minimize
  - Examples
    - Maximize profit
    - Minimize cost
    - Minimize distance
    - Minimize time
Components of LP Problem (Cont.)

- Constraints
  - Optimization condition that is represented mathematically in the model
  - Usually resource limitations
- Examples
  - Money
  - Labor
  - Time
  - Equipment

5 Steps In Formulating MP Models:

1. Understand the problem.
2. Identify the decision variables.
   \[ X_1 = \text{number of } \ldots \]
   \[ X_2 = \text{number of } \ldots \]
3. State the objective function as a combination of the decision variables.
   \[
   \text{MAX: } \ldots
   \]
4. State the constraints as combinations of the decision variables.
   \[
   \ldots
   \]
5. Identify any upper or lower bounds on the decision variables.
   \[
   \ldots
   \]

Model Formulation: Example 1

Logo-motion is a sports apparel firm that manufactures jackets, hats, sweat outfits, and T-shirts for college and professional athletic teams. It has contracted with the State University Bookstore for two types of logo jackets, a deluxe jacket and a regular jacket. The deluxe jacket is heavier, with more pockets, a nicer lining, and an embroidered school name and logo. The regular jacket has sewn-on prefabricated logos and lettering. The major steps in the manufacture of these jackets are cutting the material, sewing, and decorating with embroidery or sewn-on items. The following table shows the resource requirements for each type of jacket and total weekly availability of resources.
Formulate a linear programming model to determine how many deluxe and regular jackets the company should produce in order to maximize profit.

**Objective Function**

Maximize $Z = 18X_1 + 12X_2$

**Constraints**

- Cutting:
  
  $0.16X_1 + 0.15X_2 \leq 40$ hrs

- Sewing:
  
  $0.47X_1 + 0.28X_2 \leq 80$ hrs

- Decoration:
  
  $0.40X_1 + 0.14X_2 \leq 55$ hrs

**Nonnegativity Constraints**

$X_1 \geq 0$

$X_2 \geq 0$

OR

$X_1, X_2 \geq 0$

*State the objective as a linear combination of the decision variables*

*State the constraints as a linear combination of the decision variables*
Example 1 (Cont.)

- Complete LP Model would be written as
  \[ \text{Max } Z = 18X_1 + 12X_2 \]
  subject to:
  \[ 0.16X_1 + 0.15X_2 \leq 40 \text{ hrs} \]
  \[ 0.47X_1 + 0.28X_2 \leq 80 \text{ hrs} \]
  \[ 0.40X_1 + 0.14X_2 \leq 55 \text{ hrs} \]
  \[ X_1, X_2 \geq 0 \]

Example 2

Inform, Inc., a media marketing firm, has contracted with a company to market its product. The company wants its TV and radio advertising to reach different numbers of customers within three age-groups: over 40, between 25 and 40, and under 25 year old. One minute of TV commercial time costs $7,000 and will reach an average of 16,000 viewers in the over-40 group, 12,500 customers in the 25-to-40 group, and 8,600 in the under-25 group. One minute of radio time costs $2,500 and will reach 14,000 listeners in the over-40 age-group, 8,000 in the 25-to-40 age-group, and 14,000 in the under-25 group. The company wants to have a total exposure of 70,000 in the over-40 group, 80,000 in the 25-40 age-group, and 65,000 in the under-25 group. Formulate an LP model to determine the amount of different commercial minutes to use at the minimum cost.

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Method</th>
<th>&lt; 25</th>
<th>25 - 40</th>
<th>&gt; 40</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>Television</td>
<td></td>
<td>8,600</td>
<td>12,500</td>
<td>16,000</td>
<td>$7,000</td>
</tr>
<tr>
<td>Radio</td>
<td></td>
<td>14,000</td>
<td>8,000</td>
<td>4,500</td>
<td>2,500</td>
</tr>
<tr>
<td>Viewers</td>
<td></td>
<td>70,000</td>
<td>80,000</td>
<td>65,000</td>
<td></td>
</tr>
</tbody>
</table>

Example 2 (Cont.)

- Decision Variables
  - Let \( Z \) = Exposure Cost
  - Let \( X_1 \) = number of minutes of TV commercials
  - Let \( X_2 \) = number of minutes of Radio commercials

Example 2 (Cont.)

- Objective Function
  \[ \text{Min } Z = 7,000X_1 + 2,500X_2 \]

(subject to)

- Constraints
  \[ 8,600X_1 + 14,000X_2 \geq 70,000 \text{ (< 25)} \]
  \[ 12,500X_1 + 8,000X_2 \geq 80,000 \text{ (25-40)} \]
  \[ 16,000X_1 + 4,500X_2 \geq 65,000 \text{ (> 40)} \]
  \[ X_1, X_2 \geq 0 \]
Example 3
Farmer Bill has 300 acres and plans to plant wheat and soybeans. Each acre of wheat costs $275 to plant, maintain and harvest while each acre of soybeans costs $140 to plant, maintain and harvest. The farmer has a crop loan of $60,000 available to cover costs. Each acre of wheat will yield 120 bu. of wheat, while each acre of soybeans will yield 30 bu. of soybeans. Farmer Bill has contracted to sell the wheat and soybeans for $3.00 per bu. and $6.00 per bu., respectively. However, the farmer must store both the wheat and soybeans for several months after harvest in his storage facility which has a maximum capacity of 24,000 bu. Farmer Bill wants to know how many acres of each crop to plant in order to maximize profit. Formulate a linear programming model for this problem.

Example 3 (Cont.)
• Decision Variables
Let \( Z \) = Profit for 300 acres of crops
Let \( X_1 \) = number of acres of wheat to plant
Let \( X_2 \) = number of acres of soybeans to plant

Objective Function
Max \( Z = 85X_1 + 40X_2 \)

Example 3 (Cont.)
• Setting up a table, we have

<table>
<thead>
<tr>
<th></th>
<th>Yield/acre (bushel)</th>
<th>Cost/acre ($)</th>
<th>Profit/acre ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wheat</td>
<td>120</td>
<td>275</td>
<td>???</td>
</tr>
<tr>
<td>Soy Beans</td>
<td>30</td>
<td>140</td>
<td>???</td>
</tr>
<tr>
<td>Capacity</td>
<td>24,000</td>
<td>60,000</td>
<td>???</td>
</tr>
</tbody>
</table>

Profit/acre for Wheat:
Cost: \( \frac{275/acre}{120 \text{ bu/acre}} = 2.29 \)
Profit/bu = $3 - $2.29 = $0.7083/bu
Profit/acre = $0.7083/bu (120 bu/acre) = $85
Profit/acre for Soybeans: $40

Example 3 (Cont.)
• Constraints
\[ 120X_1 + 30X_2 \leq 24,000 \] (yield - storage capacity)
\[ 275X_1 + 140X_2 \leq 60,000 \] (costs - loan amount)
\[ X_1 + X_2 \leq 300 \] (acres - size of farm)
\[ X_1, X_2 \geq 0 \]
Example 4

Universal Stone House Association (USHA) has an advertising budget of $100,000 to spend each year on television, radio, and direct mailing. USHA has advertised using all three methods in the past and plans to spend at least 10% on each method this year as well. Further, they would like to spend at least as much on direct mailing as television and radio combined since the publisher will give them discounts on other business publishing needs. Historical data on the financial returns of each advertising method is summarized below.

Example 4 (Cont.)

Can’t really set up a table, but we can list each item of relevance:
1. Advertising budget: $100,000
2. Spend at least 10% on each technique
3. Spend at least as much on direct mailing as TV & radio combined
4. Maximize return

Example 4 (Cont.)

<table>
<thead>
<tr>
<th>Advertising Technique</th>
<th>Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>Television</td>
<td>8%</td>
</tr>
<tr>
<td>Radio</td>
<td>6%</td>
</tr>
<tr>
<td>Direct Mailing</td>
<td>6%</td>
</tr>
</tbody>
</table>

How should USHA divide the budget to meet all of the requirements and yet maximize returns? Formulate an LP model to answer this question.

Example 4 (Cont.)

• Decision Variables:
  Let \( Z = \text{Total Cash} \) (what about total return?)
  Let \( X_1 = \text{the } \$\text{amount spent for TV ads} \)
  Let \( X_2 = \text{the } \$\text{amount spent for radio ads} \)
  Let \( X_3 = \text{the } \$\text{amount spent for direct mailing} \)

• Objective Function:
  \[ \text{Max } Z = 1.08X_1 + 1.06X_2 + 1.06X_3 \]
Example 4 (Cont.)

• Constraints:
  \[ X_1 + X_2 + X_3 = \$100,000 \]
  \[ X_1, X_2, X_3 \geq \$10,000 \]
  \[ X_3 \geq X_1 + X_2 \text{ OR } X_1 + X_2 - X_3 \leq 0 \]

Graphical Solution to LP Models

• Two approaches:
  – Graphically (enumeration)
  – Mathematically (Simplex Method; Excel)

• Constraints define the feasible region
• What is the feasible region?
  set of points (or values) that the decision variables can have and simultaneously satisfy all of the constraints in the LP model.

Graphical Solution (Cont.)

• Two variables... we can graph the FR and locate the optimal solution/point
• Must first plot the constraints
• Recall that the lines are of the form \( aX_1 + bX_2 = c \)
  where \( a, b, \) and \( c \) are constants.

• Let's look at the Farmer Bill Example...

Graphical Solution: Farmer Bill Example

• Recall the model
  \[ \text{Max } Z = 85X_1 + 40X_2 \]
  subject to
  \[ 120X_1 + 30X_2 \leq 24,000 \text{ (yield)} \]
  \[ 275X_1 + 140X_2 \leq 60,000 \text{ (costs)} \]
  \[ X_1 + X_2 \leq 300 \text{ (acres)} \]
  \[ X_1, X_2 \geq 0 \]
Graphical Solution: Farmer Bill Example

• Constraint 1
  \[120X_1 + 30X_2 \leq 24,000 \text{ (yield)}\]

• Find any two points on the line and then we can draw the constraint
  Let \(X_1 = 0\) and solve for \(X_2\)
  \[30X_2 = 24,000\]
  \[X_2 = 800\]
  \((0, 800)\) is a point on the line

Graphical Solution: Farmer Bill Example

• Similarly, let \(X_2 = 0\) and solve for \(X_1\)
  \[120X_1 = 24,000\]
  \[X_1 = 200\]
  \((200, 0)\) is another point on the line
Graphical Solution: Farmer Bill Example

Now we can plot the cost constraint line along with the yield.

Lastly, we can do the same for the acreage constraint line.

Plotting all constraints...

• Infinite Number of points in feasible region
• How do we find the best/optimal one?
• FACT: It has been shown that the solution to an LP problem will always occur at a point in the feasible region where two or more of the boundary lines of the constraints intersect.
• Points are called corner points or extreme points
Graphical Solution: Farmer Bill Example

*Why Boundary Points → Solution*

• Suppose you want to find $X_1$ and $X_2$ that yields a profit of $10,000
• Mathematically, 
  
  $10,000 = 85X_1 + 40X_2$
  
• Can plot above equation
• Similarly, suppose you want to find values of $X_1$ and $X_2$ that gives us a profit of $15,000.
  
  $15,000 = 85X_1 + 40X_2$
• Can also plot this equation

Graphical Solution: Farmer Bill Example

*Why Boundary Points → Solution (Cont.)*

• Farmer Bill Example...
• Optimal solution at point C
• Intersection between Cost and Yield constraints
• Solve two equations with two unknowns
  
  $120X_1 + 30X_2 = 24,000$ (yield)
  
  $275X_1 + 140X_2 = 60,000$ (costs)
Graphical Solution: Farmer Bill Example

Obtaining the Optimal Solution

• Solve first (yield) equation for $X_1$:
  \[ X_1 = \frac{24,000 - 30X_2}{120} = 200 - 0.25X_2 \]

• Now substitute into the second (cost) equation:
  \[ 275(200 - 0.25X_2) + 140X_2 = 60,000 \]
  \[ 55,000 - 68.75X_2 + 140X_2 = 60,000 \]
  \[ 71.25X_2 = 5,000 \]
  \[ X_2 = 70.1754 \]

Obtaining the Optimal Solution (Cont.)

• Given the value of $X_2$, we can now find $X_1$:
  \[ X_1 = 200 - 0.25(70.1754) = 182.4561 \]

• Maximize profit by planting 182.5 acres of wheat and 70.18 acres of soybeans for a profit of
  \[ Z = 85(182.5) + 40(70.18) = 18,315.79 \]

Graphical Solution: Farmer Bill Example

Enumeration

• Level curves make process rely on “visual inspection.”
• Another alternative: Enumeration of corner points

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Obj. Function Value (Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (0, 300)</td>
<td>$12,000.00</td>
</tr>
<tr>
<td>B (133.33, 166.67)</td>
<td>$17,999.85</td>
</tr>
<tr>
<td>C (182.5, 70.18)</td>
<td>$18,315.79</td>
</tr>
<tr>
<td>D (200, 0)</td>
<td>$17,000.00</td>
</tr>
</tbody>
</table>

Solving a Minimization Problem

Inform, Inc. Example

Let $Z = \text{Exposure Cost}$
Let $X_1 = \text{number of minutes of TV commercials}$
Let $X_2 = \text{number of minutes of Radio commercials}$

\[ \text{Min } Z = 7,000X_1 + 2,500X_2 \]
subject to
\[ 8,600X_1 + 14,000X_2 \geq 70,000 \quad (< 25) \]
\[ 12,500X_1 + 8,000X_2 \geq 80,000 \quad (25-40) \]
\[ 16,000X_1 + 4,500X_2 \geq 65,000 \quad (> 40) \]
\[ X_1, X_2 \geq 0 \]
Solving a Minimization Problem
Inform, Inc. Example
• Level curves go toward the origin
• Minimum point will be closest to the origin
• Enumeration...

<table>
<thead>
<tr>
<th>Corner Point</th>
<th>Obj. Function Value (Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A (0, 14.44)</td>
<td>$36,111.00</td>
</tr>
<tr>
<td>B (2.23, 6.52)</td>
<td>$31,899.25</td>
</tr>
<tr>
<td>C (5.27, 1.76)</td>
<td>$41,313.70</td>
</tr>
<tr>
<td>D (8.14, 0)</td>
<td>$56,980.00</td>
</tr>
</tbody>
</table>

Special Conditions in LP Models
• An exact optimal solution will not always exist
• Can have the following situations:
  – Multiple Optimal Solutions
  – No Solution
  – Unbounded
• How can we identify these?

Inform, Inc. Plot

Multiple Optimal Solutions
• Alternate Optimal Solutions
• More than one optimal solution to the LP problem
• Example
  Max Z = 40X₁ + 30X₂
  subject to
  X₁ + 2X₂ ≤ 40
  4X₁ + 3X₂ ≤ 120
  X₁, X₂ ≥ 0
Multiple Optimal Solutions (Cont.)

- Optimal solution at corner points
- Level curve (objective function) intersects BC line segment before leaving the feasible region
- Every point on BC is optimal
- Gives decision maker more alternatives and greater flexibility

Multiple Optimal Solutions (Cont.)

- Objective Function
  \[ X_2 = -\frac{4}{3}X_1 + \frac{Z}{30} \]
- Constraints
  \[ X_2 = -\frac{1}{2}X_1 + 20 \]
  \[ X_2 = -\frac{4}{3}X_1 + 40 \]

Multiple Optimal Solutions (Cont.)

Infeasible Problem

- No feasible solution area by the constraints
- Example
  \[ \text{Max } Z = 5X_1 + 3X_2 \]
  subject to
  \[ 4X_1 + 2X_2 \leq 8 \]
  \[ X_1 \geq 4 \]
  \[ X_2 \geq 6 \]
  \[ X_1, X_2 \geq 0 \]
**Unbounded Problem**

- Feasible region is not closed, which allows the objective function to increase indefinitely
- Example
  \[
  \text{Max } Z = 4X_1 + 2X_2 \\
  \text{subject to} \\
  X_1 \geq 4 \\
  X_2 \leq 2 \\
  X_1, X_2 \geq 0
  \]

**Steps to Solving an LP Model using MS Excel**

1. Organize the data for the model on the spreadsheet.
2. Reserve separate cells in the spreadsheet to represent each decision variable in the model.
3. Create a formula in a cell in the spreadsheet that corresponds to the objective function.
4. For each constraint, create a formula in a separate cell in the spreadsheet that corresponds to the left-hand side (LHS) of the constraint.
**Logo-motion Example using Excel**

Max Z = $18X_1 + $12X_2 \quad \text{Profit}
subject to
0.16X_1 + 0.15X_2 \leq 40 \text{ hrs} \quad \text{Cutting}
0.47X_1 + 0.28X_2 \leq 80 \text{ hrs} \quad \text{Sewing}
0.40X_1 + 0.14X_2 \leq 55 \text{ hrs} \quad \text{Decoration}
X_1, X_2 \geq 0

---

**Sumproduct Statement (Cont.)**

Example:
Suppose you want to multiply the data in columns A and B and then add them.

\[ 3 \times 4 + 8 \times 6 + 1 \times 9 + 2 \times 7 + 6 \times 7 + 5 \times 3 = 140 \]

In Excel, one would write the statement:

\[ = \text{Sumproduct}(A1:A6,B1:B6) \]

---

**Useful Excel Statement**

- **SUMPRODUCT**\text{(array1, array2, array3, ...)}
- array1, array2, array3, ... are 2 to 30 arrays whose components you want to multiply and then add.
- Array arguments must have the same dimensions.

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**Implementing The Model**

[Excel spreadsheet image]

Max Z = $18X_1 + $12X_2 \quad \text{Profit}
subject to
0.16X_1 + 0.15X_2 \leq 40 \text{ hrs} \quad \text{Cutting}
0.47X_1 + 0.28X_2 \leq 80 \text{ hrs} \quad \text{Sewing}
0.40X_1 + 0.14X_2 \leq 55 \text{ hrs} \quad \text{Decoration}
X_1, X_2 \geq 0
How Solver Views the Model

• Target cell - the cell in the spreadsheet that represents the *objective function*
• Changing cells - the cells in the spreadsheet representing the *decision variables*
• Constraint cells - the cells in the spreadsheet representing the *LHS formulas* on the constraints

Solver in Excel...

• Select Solver… in Data Analysis Section
• If Solver… is not visible in Data Analysis section, then
  – Select Add-Ins from the Excel Options
  – Then, scroll down the list of Add-Ins Available in the Add-In Manager until you see Solver Add-In
  – Check the corresponding box
  – Click OK
**Solver Options...**

**Goals For Spreadsheet Design**

- **Communication** - A spreadsheet's primary business purpose is that of communicating information to managers.
- **Reliability** - The output a spreadsheet generates should be correct and consistent.
- **Auditability** - A manager should be able to retrace the steps followed to generate the different outputs from the model in order to understand the model and verify results.
- **Modifiability** - A well-designed spreadsheet should be easy to change or enhance in order to meet dynamic user requirements.

**Solver Results...**

**Spreadsheet Design Guidelines**

- Organize the data, then build the model around the data.
- Do not embed numeric constants in formulas.
- Things which are logically related should be physically related.
- Use formulas that can be copied.
- Column/rows totals should be close to the columns/rows being totaled.
- Use color, shading, and borders to distinguish changeable parameters from other model elements.
- Use text boxes and cell notes to document various elements of the model.